Vertex Coloring of Planar Graphs with Restrictions over Shortest Paths

Faculty Advisor:

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Abstract:

We define a Shortest-Path Square-Free (SPSF) graph coloring to be a vertex coloring such that between any two vertices of the graph, there is a shortest path which is Square-Free in the sense defined by Thue. We will investigate the conjecture that every planar graph requires no more than four colors in order to generate a SPSF coloring.

Description:

<u>The Four-Color Theorem</u>**:** Every planar graph has a proper vertex coloring using no more than 4 colors.

In 1976 by Kenneth Appel and Wolfgang Haken proved the theorem stated above and solved a problem which had been formalized nearly one hundred years earlier. For centuries, cartographers had noted that no more than four colors seemed to be necessary to color maps so that no two adjacent regions had the same color. Nineteenth century mathematicians hypothesized that no more than four colors would be needed for any conceivable map. With the development of Graph Theory, this hypothesis was rephrased in the language of planar graphs.

In 1890, Heawood found a simple that no more than five colors were necessary to color a planar graph so that no two adjacent vertices had the same color. For the next eight decades, the truth of the Four-Color Conjecture remained unsettled. Both attempts to produce a graph requiring five colors for a proper coloring, and attempts to reduce the bound in Heawood's theorem to four met with no success.

By the 1970s, the Four-Color problem had gained mathematical celebrity among the general public second only to Fermat's Last Theorem. The Haken and Appel (HA) proof caused a splash by settling this long open problem, but at the same time it set off a new controversy.

Obviously, since there are infinitely many possible maps, the HA proof could not proceed by checking them all directly. Rather, the first step of the proof was a reduction to a finite number of cases to check. Such a reduction is common in mathematical proofs. Often, one is able to reduce the number of cases to two or three. The HA proof reduced the number of required cases to 1936. Checking each of these cases by hand would have been impractical. A computer program was written to verify each of these cases. The program was run, the cases were checked, and the proof was complete.

The use of a computer program as part of a mathematical proof was a new idea in 1976. The HA proof touched off debate as to what should qualify as a proof in mathematics. While we might be able to verify that the Four-Color Theorem is true, the fact that the proof is so complex means that our understanding of the problem is far from complete. With few exceptions, the simpler the proof, the more highly it is regarded.

As a result, since 1976, many attempts have been made to find a simple proof of the Four Color Theorem. While some simplifications of the HA proof have been made, currently no proof is known which is short enough to be verified by a single person without computer aid. In 2007, Janos Barat and Peter Varju defined a square-free vertex coloring on a graph to be a vertex coloring such that along paths in the graph, there is no sequence of vertices with a color sequence containing two adjacent, identical blocks. Clearly, as square-free vertex coloring is also a proper vertex coloring, and so the required number of colors for a square-free coloring is at least as big as the number of colors required for a proper vertex coloring.

The concept of a square-free vertex coloring does not seem to be of much help in shedding light on the Four-Color Theorem, since, with very little effort, one can construct planar graphs requiring at least five colors in order to construct a square-free vertex coloring.

This summer, I would like to investigate a variation on this definition:

We define a shortest-path square-free (SPSF) coloring to be a vertex coloring for which for each pair of vertices in the graph, along some shortest path between them is colored such that there are no adjacent, identical blocks of colors. This is also a generalization of a proper graph coloring. So far, I have not found a planar graph requiring more than four colors in order to construct a SPSF coloring.

Conjecture:

At most four colors are necessary to construct a SPSF coloring of a planar graph.

If the conjecture is true, it would imply the Four Color Theorem.

Zhe will begin by manually constructing SPSF colorings for small planar graphs. He will proceed to develop programs to determine the minimum number of colors required to construct an SPSF coloring by the greedy algorithm.

This project will proceed in one of two ways. Either we will discover a counterexample to the conjecture, or we will not. If we do, we will have to modify the definition of SPSF and modify the conjecture accordingly. If not, we may try to prove this conjecture. In all honesty, even if the conjecture is true, it is likely to be very difficult to prove. However, a counterintuitive as it may be, sometimes in mathematics the proof of a stronger result is actually easier than the proof of a weaker one.

Of the two possible outcomes, the lack of a counterexample would be, by far, the more interesting. Such an outcome might suggest a new approach to the Four-Color Theorem and be of interest to various groups currently working on a simpler proof.

Dissemination Goals:

Certainly, Zhe will want to present the results of his work at MARCUS.
My past summer students have presented their work successfully at that conference. Depending on the results of our work groups led by J. Ferro, G. Gonthier and others might be interested in our approach.

External Funding: None.

Budgetary Needs:

Travel (to library at UVA) $100
Office Supplies $50

Past Outcomes: During summer of 2010, I conducted research with Guan Wang which continued as a Senior Honors Project. We did, in fact, prove a theorem and are in the process of preparing a paper for publication.

Student Researcher: I plan to structure my working relationship with Zhe after the one I had with Guan Wang. I will meet with him two to three times a week. Zhe will alternate between experimenting with pencil and paper, and writing computer programs to aid our search. Increasingly, computer experimentation is becoming an accepted part of mathematical research, but most undergraduate mathematics students get almost no exposure to this sort of activity. Based on my interaction with Zhe in my Six Games course, he has facility for this kind of algorithmic approach.

Student Statement:

Last semester, I took Dr. Irwin's Math 337-Elementary Number Theory-as my beginning in my college life. It's rare that a fresh man would take such a high level class and I was also surprised about my own decision. But this is me, who was absorbed by the kingdom of those fantastic graphs and numbers since I was a young boy. The project I am going to research with Dr. Ordower is Four Color Theorem, which is an old math problem, dating to over four thousand years ago. I have been fascinated by this project for a long time, since I came into contact with this puzzle when I was in junior high school, and till now I still express considerable interests in this topic. What this problem generally talks about is to color a map using four colors and no adjacent two countries are of the same one, but the difficulty and the practical significance of this problem is far beyond people's imagination. That is the main reason why I like this problem since its limitless particle application. Compared to algebra, I prefer geometry; no complicated

formula, no arcane concept, all you have to do is jumping in the puzzle to seek for the secrets behind the graphs.
Since I have learned some basic knowledge on this project in my junior high school, and I am confident with my ability of finding rules from mathematical graphs due to my consistent training on these kinds of problem in my spare time. Most importantly, this project suited to my preference and math major perfectly. I cannot even wait until May to start studying this problem! I always believe that passion is the most important factor when studying something. I wish to engage my intelligence and delve into this project to discover something behind this problem. There is no other more splendid thing than “playing” with math for 8 weeks without caring about my grade. Working with Dr. Ordower will broaden my vision on different methods of resolving a math problem. Moreover, working on such a world known puzzle will absolutely become an impressive experience for me. Not only can I improve my math skills, but also I can be more confident studying math in the next four years.